Supplementary Material: Extension to Spherical Budget Zones in $\mathbb{R}^3$

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1 Extension to Spherical Budget Zones in $\mathbb{R}^3$

The bounded-deviation theorem and the resulting narrow-corridor property are stated and proved for planar circular budget zones. We then show that both the theorem and the IRD algorithm extend directly to spherical budget zones in three-dimensional space. Crucially, this extension does not require new inductive arguments; it relies only on the structure of the base case.

Dependence of the Bounded-Deviation Proof on the Base Case. Examination of the proof of the bounded-deviation theorem (from the main paper) reveals that the induction over the number of budget zones relies on a single geometric property of the base case ($n = 1$): the total path length is a continuous function of the boundary entry (and exit) location. No step of the induction exploits planarity or circle-specific geometry beyond this continuity property.

In the planar case, the base-case objective is a continuous function of a single angular parameter defining the entry point on the circle. Continuity implies that the discretized optimum must select a boundary point within arc-length distance at most $d/2$ from the continuous optimum. This approximation property is the only ingredient required to initiate the inductive argument.

1.1 Single-Sphere Base Case in $\mathbb{R}^3$

We make the base case ($n = 1$) explicit for a single spherical budget zone in three dimensions. Let

$$B = \{X \in \mathbb{R}^3 : \|X - o\| \leq r\}$$

be a sphere of radius $r > 0$ centered at $o \in \mathbb{R}^3$, and let $s, t \in \mathbb{R}^3 \setminus B$. Fix an in-sphere budget $b \in (0, 2r)$, measured as the Euclidean length of the straight chord traversed inside the sphere. Any feasible path of the form $\Pi = s \to P \to Q \to t$ with $P, Q \in \partial B$ must satisfy the chord constraint $\ell(P, Q) = b$, where for points $X, Y \in \mathbb{R}^3$ we write $\ell(X, Y) = \|X - Y\|$.

Parameterization of the entry point. We parameterize the entry point $P \in \partial B$ by spherical angles $(\alpha, \beta) \in [0, 2\pi] \times [0, \pi]$ via

$$P(\alpha, \beta) = o + r\, u(\alpha, \beta), \qquad u(\alpha, \beta) = \begin{pmatrix} \cos\alpha \, \sin\beta \\ \sin\alpha \, \sin\beta \\ \cos\beta \end{pmatrix} \tag{1}$$

Chord geometry and construction of the exit point. The in-sphere segment PQ is a chord of length b, hence it subtends the (fixed) central angle

$$\theta = 2\arcsin\left(\frac{b}{2r}\right)$$

Define the unit direction from the sphere center toward the target as

$$v = \frac{t-o}{\|t-o\|}$$

An optimal chord lies in the plane spanned by $u(\alpha,\beta)$ and v; otherwise, rotating the chord about the axis through o and P preserves feasibility while increasing $\|Q-t\|$ by symmetry. Let

$$w = \begin{cases} \dfrac{u\times v}{\|u\times v\|} & \text{if } u \not\parallel v \\ \text{any unit vector orthogonal to } u, & \text{otherwise} \end{cases}$$

Using Rodrigues' rotation formula, the exit direction is obtained by rotating u by angle θ about axis w:

$$\tilde{u} := u\cos\theta + (w\times u)\sin\theta + w(w\cdot u)(1-\cos\theta)$$

and therefore the exit point is

$$Q(\alpha,\beta) = o + r\,\tilde{u}(\alpha,\beta) \tag{2}$$

Since $P = o + r\,u$ and $Q = o + r\,\tilde{u}$ lie on the sphere and the rotation preserves the central angle $\angle(u,\tilde{u}) = \theta$, the chord-length formula

$$\ell(P,Q) = 2r\sin(\theta/2) = b$$

gives

$$\ell(P(\alpha,\beta),Q(\alpha,\beta)) = b$$

Base-case objective. The corresponding objective function is

$$f(\alpha,\beta) := \ell(s,P(\alpha,\beta)) + b + \ell(Q(\alpha,\beta),t)$$

Base Case for a Single Spherical Budget Zone. For a single spherical budget zone in $\mathbb{R}^3$, any feasible path with fixed in-sphere budget b can be parameterized by two angular parameters (α,β) defining the entry point on the sphere boundary. The corresponding objective function is

$$f(\alpha,\beta) = \ell(s,P(\alpha,\beta)) + b + \ell(Q(\alpha,\beta),t),$$

where $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ are defined as in Equations 1, 2.

Lemma 1 (Continuity of the Spherical Base-Case Objective). *The function $f(\alpha, \beta)$ is continuous on the domain*

$$(\alpha, \beta) \in [0, 2\pi] \times [0, \pi]$$

and attains a global minimum.

Proof. The maps $(\alpha, \beta) \mapsto P(\alpha, \beta)$ and $(\alpha, \beta) \mapsto Q(\alpha, \beta)$ are continuous (trigonometric parameterization and a fixed-angle rotation), hence f is continuous as a sum of norms and a constant. ∎

Implications for Bounded Deviation and IRD. The bounded-deviation induction for circles used only this base-case approximation property, so it extended verbatim to spherical zones. Consequently, IRD and its refinement strategy applied unchanged in $\mathbb{R}^3$.

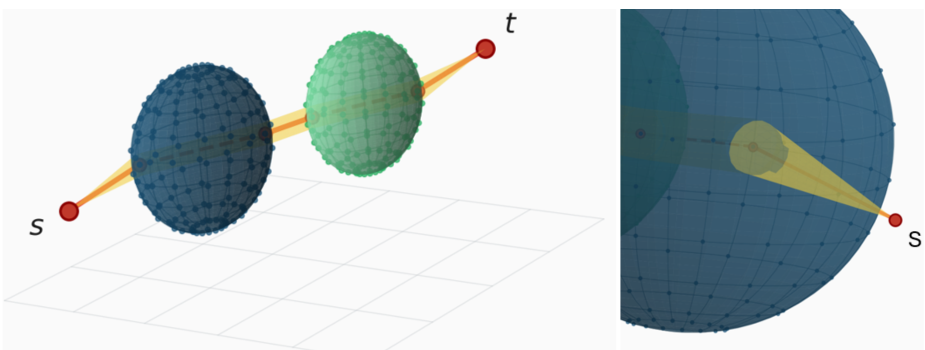

Fig. 1: Illustration of the spherical budget zone extension in $\mathbb{R}^3$.